Homework Assignment #1: Answer Sheet

1. Suppose there is no uncertainty and that marginal utility in any period is given by $u'(c_t) = c_t^{-\gamma}$. The basic asset price equation is given by $p_t = E_t[m_t x_t]$. So we can write the gross risk-free rate as $R_f = \frac{1}{E(m_{t+1})}$. Show that this implies that $R_f = \frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\gamma}$ where R_f is the risk free gross interest rate. Explain this expression. How does R_f vary with β and γ? Use excel to solve for R_f with $\beta = .9$ and $\gamma = 2$. Let consumption growth be 2%.

   brief answer *We have $R_f = \frac{1}{E(m_{t+1})}$, and no uncertainty so*

   $$m_{t+1} = \beta\frac{u'(c_{t+1})}{u'(c_t)} = \beta\frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}},$$

   so $R_f = \frac{1}{\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}}$ or $R_f = \frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\gamma}$. The expression tells us that the gross risk-free rate is: (i) high when β is low; (ii) that it is high when consumption growth is high; (iii) that rates are more sensitive to consumption growth when γ are high, that is when agents are more risk averse. Notice that if $\gamma = 0$ then the interest rate is independent of consumption growth. This makes sense, because $\gamma = 0$ implies that agents do not prefer smooth consumption– their indifference curves are straight lines in that case. $\frac{1}{.9}*(1.02)^2 = 1.156$.

   (a) Suppose that γ is now 4, what happens to the risk-free rate?

   brief answer *$\frac{1}{.9} * (1.02)^4 = 1.2027$, the risk-free rate rises from 15.6% to 20.2%.*

   (b) Suppose instead that $\beta = .8$ what happens to the risk-free rate?

   brief answer *$\frac{1}{.8} * (1.02)^2 = 1.3005$, the risk-free rate rises.*

   (c) Suppose that consumption growth rises to 5%, how do you answers change?

   brief answer *$\frac{1}{.9} * (1.05)^2 = 1.225$, so faster consumption growth is associated with a higher interest rate. Also, $\frac{1}{.9}*(1.05)^4 = 1.3506 > 1.20$, $\frac{1}{.8}*(1.05)^2 = 1.3781 > 1.300$. This should be intuitive: faster consumption growth $\left(\frac{c_{t+1}}{c_t}\right)$ rising means more current savings. It makes sense that you save more in this case.*

   (d) Provide intuition to explain why the risk-free rate responds as you describe.

   brief answer *some is already provided in (c). Note that higher γ implies more risk aversion so agents need higher compensation to have non-smooth consumption. And if people are more impatient, $\beta \uparrow$ then it takes a higher interest rate to get people to postpone consumption. And if $\frac{c_{t+1}}{c_t} > 1$ that is exactly what they are doing.*

2. Consider a portfolio, c, comprised of a risk-free and risky asset, with returns given by r_f and $E(r_p)$, respectively. Let y be the proportion of the portfolio invested in the risky asset. Let the utility function of the agent be given by

$$U = E(r) - \frac{1}{2}A\sigma^2 \qquad (1)$$

where A is a constant.

(a) Why does (1) make sense? What does it mean if $A < 0$? If $A = 0$? Draw the indifference curves of an agent who has the preferences described by (1) for each of the three cases: $A < 0, A = 0, A > 0$. Which case is the most sensible?

brief answer *Utility depends on positively on expected return and negatively on the risk. This makes sense if people are risk averse or risk loving. Indeed, this equation can even represent preferences for risk-neutral agents ($A = 0$). In the latter case indifference curves will have zero slope: the agent cares only about expected return, and is indifferent to risk. For $A > 0$ indifference curves will be positively sloped: the agent is risk averse, implying that higher expected return is required to get her to hold more risk. If $A < 0$ the indifference curves are negatively sloped as the agent is a risk lover. $A > 0$ makes more sense given that agents typically purchase insurance.*

(b) Denote the standard deviation of the risky asset as σ_p. What will the standard deviation of the portfolio, σ_c, be equal to? What will the variance of the portfolio, σ_c^2, be equal to?

brief answer *The risk-free asset must have zero variance. So $\sigma_c = y\sigma_P$. Then $\sigma_c^2 = y^2\sigma_c^2$.*

(c) What will the expected return on the complete portfolio, $E(r_c)$ be equal to?

brief answer *A weighted average of the risk-free rate and the expected return on the portfolio, $E(r_c) = yE(r_P) + (1 - y)r_f$.*

(d) How will the optimal choice of y vary with (i) the variance of the risky asset; (ii) the excess return on the risky asset; (iii) the value of A?

brief answer *If σ_P^2 increases, then y will decrease given $A > 0$; this is the case of the capital allocation line becoming flatter. If $E(r_P) - r_f$ increases then y will increase; this is equivalent to the capital allocation line becoming steeper. If A rises then the agent is more risk averse, so he will hold less y. Think of it as the indifference curve becoming steeper.*

3. Suppose firm has assets in place which pay off next period depending on which of the two possible states occurs:

$$\begin{array}{lll} boom & payoff = 100 & \text{with } prob = .65 \\ bust & payoff = 20 & \text{with } prob = .35 \end{array}$$

Assume everyone is risk neutral, that $r = 0$, and there are no taxes. Assume the firm has debt with face value, $F = 50$. What is the value of the firm, V equal to?

brief answer $V = .65 * 100 + .35 * 20 = 72.$

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(a) What is the value of the equity and debt of this firm?

brief answer *Debt is the residual claimant, so he payoffs for debt and equity are given by:* $D = \min(V, F)$, $E = \max(V - F, 0)$. *So*

$$\begin{array}{ll} boom & D = F = 50, E = 50 \\ bust & D = V = 20, E = 0 \end{array}$$

The values today depend on the expected values, and since the states occur with probabilities $(.65, .35)$, *we must have:* $D = .65 * 50 + .35 * 20 = 39.5$, *and* $E = .65 * 50 = 32.5$. *The value of the firm is* $D + E = 72$. *This means that the debt is under water, it is trading below par, since* $D = 39.5 < 50 = F$.

(b) Now suppose that the firm is faced with new investment opportunity that will create profits for sure. If it invests 10 today, the firm will receive 12 tomorrow in either state. What happens to the firm's cash flow and values in both states if it takes on the investment? What happens to the firm's value?

brief answer *Now the payoffs in each state are given by*

$$\begin{array}{ll} boom & \text{cash flow equals } 100 + 12 = 112 \\ bust & \text{cash flow equals } 20 + 12 = 32 \end{array}$$

so the firm's value increases from 72 to 84, since $112 * .65 + .35 * 32 = 84.0$.

(c) Will equity holders invest the 10 today in this investment?

brief answer *No. Even though* $V = 84$, *the issue is how is the value split between equity and debt. We now have:*

$$\begin{array}{ll} boom & D = 50, E = 62 \\ bust & D = 32, E = 0 \end{array}$$

so today the value of $D = .65 * 50 + .35 * 32 = 43.7$ *and* $E = 62 * .65 = 40.3$. *Both have increased but equity holders will not invest 10, since they would lose 2* $(= 40.3 - 32.5 - 10 = -2.2)$ *in value. There is a wealth transfer from the equity holders to the debt holders if they invest. Debt is senior and gets part of the surplus. There is a debt overhang that prevents a profitable investment from taking place. The problem is that cash flows are insufficient to cover debts in the bad state even after the investment. Notice this is arising because we have risky debt, whose value is affected by the investment. If we had risk free debt it would not happen.*

(d) Can the investment be financed by issuing debt?

brief answer *Not if it is junior to the existing debt. Then the same problem happens. Part of the surplus is transferred to the old debt. If we could issue new debt that was senior – risk free – then it is no problem. In this case:*

$$\begin{array}{ll} boom & \text{cash flows} = 112. \text{ New } D = 10, \text{ old } D = 50, E = 52 \\ bust & \text{cash flows} = 32. \text{ New } D = 10, \text{ old } D = 22, E = 0 \end{array}$$

so $E = .65 * 52 = 33.8$ *and old* $D = .65 * 50 + .35 * 20 = 39.5$, *nobody is worse off. But this does not happen. Why? Typically, there are covenants that prevent the firm from issuing senior debt. This is insisted on by bondholders to prevent the value of*

their debt from being reduced in value. But if the new debt is junior, as is always the case, then

$$boom \quad \text{cash flows} \; = \; 112. \; \text{junior } D = 10, \; \text{old } D = 50, E = 52$$
$$bust \quad \text{cash flows} \; = \; 32. \; \text{junior } D = 0, \; \text{old } D = 32, E = 0$$

the junior debt is worth $6.5 < 10$, so bad investment, and no junior debt will be issued.

(e) What if the investment returned 18 in the good state and 8 in the bad state, would your answer to part (c) change? How?

brief answer *Now things change.*

$$boom \quad \text{cash flow equals } 100 + 18 = 118$$
$$bust \quad \text{cash flow equals } 20 + 8 = 28$$

so $\dfrac{boom \quad D = 50, E = 68}{bust \quad D = 28, E = 0}$ *, so the equity holders receive $.65 * 68 = 44.2$. With the greater payoff to equity holders in the good state now equity holders will invest since $44.2 - 32.5 - 10 = 1.7 > 0$, the equity holders are better off.*