Econ 452 Professor Ickes

Economics of the Financial Crisis Spring 2014

Homework Assignment #3: Answer Key

1. Consider a model with three periods, $0, 1, 2$, and an infinite number of *ex ante* identical agents (to make life simple think of the agents as individual points on the continuum from $[0, 1]$ and normalize the entire set of agents as equal to 1). Agents learn whether they are patient or impatient in period 1. Let λ be the probability that an agent is impatient, and they choose x and y in period 0 to maximize their expected utility. If they are impatient they consume c_1 in period 1, and if they are impatient they consume c_2 in period 2. Agents are endowed with one unit of the good which they can use to purchase, in period 0, a long-lived asset (x) or a short-lived asset (y), thus, $y + x \leq 1$. The long-lived asset pays a return, $R > 1$. The short-lived asset returns one unit for one unit. In period 1 there is a market where the long asset can be sold. The price of x in period 1 is P.

   (a) Write the budget constraint for a patient and an impatient agent. Write the expression for expected utility. Show that in equilibrium $P = 1$ [Hint: Show that supply cannot equal demand if $P \neq 1$]. Given that $P = 1$, what is the market allocation? Show that this is better than what is achievable under autarky.

   brief answer *If the agent is impatient her budget constraint is* $c_1 = y + Px$, *and if she is patient then* $c_2 = \left(x + \frac{y}{P}\right) R$. *Since* λ *is the probability of being an impatient agent, expected utility is* $\lambda u(y + Px) + (1 - \lambda) u \left[\left(x + \frac{y}{P}\right) R\right]$. *Why must* $P = 1$? *Suppose not. If* $P > 1$, *then the long asset dominates the short asset at date* 0. *Nobody would hold* y. *Agents who turn out impatient will thus have to sell* x *in period one, but nobody holds* y, *so there will be no buyers. There will be an excess supply of* x *and the price* $\rightarrow 0$, *which contradicts* $P > 1$. *Now suppose that* $P < 1$, *then nobody holds* x *in period* 0. *Why not? Because you could buy it in period one for* $P < 1$, *and get more* x, *and thus more consumption in period* 2. *Late consumers want to buy the asset and earn* $\frac{R}{P} > R$, *so they will bid up the price of* P *until* $P = R > 1$, *another contradiction.*

   (b) Given an infinite number of agents a social planner can treat λ as the proportion of impatient agents in the economy. The planner wants to maximize social welfare. Write down the planners' resource constraints and expected welfare.Write down optimal choices of c_1 and c_2. If agents are risk averse will the planners' solution coincide with the market solution? Explain. [you can use graphs here]

   brief answer *The planners' resource constraint is* $\lambda c_1 + (1 - \lambda) c_2 \leq Rx + y$. *His objective is to maximize expected welfare,* $\lambda u(c_1) + (1 - \lambda) u(c_2)$. *Note that it can never be optimal for the planner to hold excess of the short asset. That excess could have been invested in the long asset and would add to second period consumption, since* $R\varepsilon - \varepsilon = (R - 1)\varepsilon > 0$. *So it follows that* $\lambda c_1 = y$, *and thus* $(1 - \lambda) c_2 = Rx$, *so* *optimal solution involves:*

$$c_1 = \frac{y}{\lambda}$$
$$c_2 = \frac{Rx}{1-\lambda}$$

(c) Define notional consumption as $C = c_1 + c_2$. Show that C^{market} in the market solution is greater than $C^{efficient}$ in the efficient solution when agents are risk averse. How can this result hold if the efficent solution is preferred to the market solution? Explain.

brief answer *The market solution* $(c_1, c_2) = (1, R)$, *so notional consumption is* $1 + R$. *In the efficient solution* $c_1 > 1$ *since* $\lambda < 1$. *For simplicity, write* $c_1 = 1 + \Delta c_1$. *But this means that* c_2 *falls by* $\Delta c_1 R$, *the foregone earnings. So in the efficient solution* $c_1 + c_2 = 1 + R + \Delta c_1 - \Delta c_1 R = 1 + R + \Delta c_1(1 - R) < 1 + R$. *Notional consumption is smaller in the efficient solution but agents are better off because they are risk averse. They are willing to pay to be insured against the event that they turn out impatient.*

(d) Explain how a bank could offer a demand deposit contract that coincides with the efficient solution.

brief answer *Let agents deposit their endowment with the bank in period* 0, *and suppose the bank holds a portfolio with* $x + y \leq 1$. *Let the bank offer a contract that allows impatient agents to withdraw* c_1 *in period one or* c_2 *in period two. If there is free entry into banking then the bank has to offer a contract that maximizes the expected utility of the agent. Notice that the bank has a first period budget constraint of* $\lambda c_1 \leq y$ *and a second-period constraint of* $(1 - \lambda)c_2 \leq Rx$. *Then the bank faces the same constraints as the social planner. So the demand deposit contract can reproduce the first-best outcomes – the efficient solution. And it is an equilibrium.*

(e) Suppose that the bank can liquidate the long asset in period one with return, $r \leq 1 < R$. Show that a bank run can be an equilibrium in this model.

brief answer *Early liquidation results in a loss equal to* $R - r$, *per unit. Suppose the banks has to liquidate sufficient* x *to meet all withdrawals that may occur in period one. If the bank liquidates all its assets it has* $rx + y \leq 1$. *If all patient agents demand early then* $c_1 > rx + y$. *We know this because* $c_1 > 1$, *and* $rx + y \leq 1$. *Thus the bank is insolvent. It has insufficient resources to meet all demands. Anyone who waits for period two gets nothing. So if an agent believes that all other agents will withdraw early, their best response is also to withdraw early. We can illustrate payoffs with the columns representing the actions of all other agents and the rows being the decision of the agent at hand.*

	run	no run
run	$rx + y, rx + y$	c_1, c_2
no run	$0, rx + y$	c_1, c_2

where $0 < rx + y < c_1 < c_2$. *There are two equilibria,* (run, run) *and* (*no run, no run*).

(f) Why is the sequential service constraint important to generate a bank run?

brief answer *The sequential service constrain induces people to get to the front of the line. If the bank could suspend withdrawals in the face of a panic, or if it could pay out less to impatient agents then patient agents may not run. But if each agent can withdraw c_1 in its entirety, then there will be insufficient funds to meet all comers in a run. Only the people at the beginning of the line will get their funds. Suppose there were n agents, then if the bank said that everybody will get c_1/n, there would be be no need to queue. But if each agent is served sequentially, and we know that the bank has less than nc_1 then it is important to get to the front of the line. That is what causes the patient agents to run.*

2. Suppose we create an asset backed security (ABS) with five mortgages. These mortgages either pay off or default, and the probability of a default is .1. Defaults are independent across the mortgages. Now suppose that we create five tranches (*senior1, senior2, senior3, mezzanine*, and *equity*). The *senior1* tranche defaults only if all five mortgages default, and the equity tranche defaults if any mortgage defaults.

   (a) Calculate the probability of default for each of the five tranches. How does the likelihood of a tranche defaulting compare with the risk of the underlying mortgages? [Note that you need to calculate the probability that, say, any two (or three, or four, etc.) of five mortgages default. This requires use of the binomial distribution, and you could use Excel or a similar program to aid your computation.]. What does this say about the risk of senior tranches?

   brief answer *The senior tranche defaults only if all five mortgages default, the senior2 if there are four defaults, and so forth. So we use the cumulative binomial distribution to calculate the ways that four mortgages default out of five experiments, then how three can default, etc. For example, the senior1 defaults only if all five default, so calculate the cumulative probability of only four defaults. Call this* BinomialDist$(4; 5, .1)$. *Then* $1 -$ BinomialDist$(4; 5, .1)$ *gives the probability of five defaults. We do this for each tranche. We obtain:*

tranche	**probability of default**
senior1	.00001
senior2	.00046
senior3	.00856
mezzanine	.08146
equity	.40951

   (b) Suppose that each mortgage was worth \$100,000, so the total pool is \$500,000. If the price of a tranche is equal to its expected value, price the senior1 tranche and the equity tranche.

   brief answer *The expected value of the senior1 tranche* $= (1 - .00001)(100,000) + (.00001)(0) = \$99,999$. *The equity tranche has expected value of* $(1 - .40951)(100,000) = 59,049$.

   (c) Suppose we now form a new security made up of mezzanine tranches. That is, we combine five securities with the same probability of default you calculated for the mezzanine tranche in part a. Call this a *CDO*. Again tranche this new security into five parts with the same pattern of seniority. Calculate the probability of default of the various tranches of the CDO.

brief answer *The probability of default of the mezzanine tranche is* .08. *So just imagine we have five bonds with the default probability of this mezzanine trance and form a new security. It is evident that the default probabilities should fall, since* $.08 < .1$, *that is the default probability of the mezzanine tranche is less than that of the underlying mortgage. So with uncorrelated risks, the CDO should be even less risky. That is what we indeed see. If we tranche it just as in part (a), and perform the same calculation we obtain:*

Default Probability of the CDO of Mezzanine Tranches

tranche	**probability of default**
senior1	.000004
senior2	.000205
senior3	.004756
mezzanine	.056117
equity	.345918

note that the senior tranches still show very low default rates.

(d) Suppose that the probability of default of the underlying mortgages is really .15. How does this change the probability of the default of the tranches of the CDO? How much riskier (say, in percentage terms) does the mezzanine tranche of the CDO get given this 50% increase in the default probability?

brief answer *With a higher underlying default probability the default probabilities of the tranches rise, and the lower tranches somewhat significantly. The default probability of the mezzanine rises from* .08146 *to* .1648. *So using*.1648 *in our CDO we get*:

Default Probability of the CDO of Mezzanine Tranches with .1648 probability of default

tranche	**probability of default**
senior1	.00012
senior2	.00314
senior3	.03397
mezzanine	.19111
equity	.59165

We see that the tranches of the CDO are now riskier that before, essentially by an order of magnitude. For example, the senior3 tranche had a default probability of .0047, now it has a default probability of .03397, almost ten times more likely. So the CDO and CDO^2 *are more sensitive to changes in the underlying default probability than the original ABS is.*

(e) What if there were 100 mortgages, 10 to a tranche, and the probability of default of the underlying loans is .05. Consider tranche10, which defaults if 10 or more mortgages default. What is the default probability of that tranche? What of the CDO made up of tranche10 securities? What happens if the underlying probability of default rises to .06?

brief answer *The analysis is the same as before. We need to calculate the likelihood that ten mortgages default out of 100 possibilities. So again use the binomial distribution. We calculate* $1 - \text{BinomialDist}(9; 100, .05) = 0.02818\,8$. *Now if we form a CDO made up to tranche10 securities we expect that these will have low default* *probabilities, since the default probability of tranche10 is less than that of the underlying mortgages* $(.028 < .1)$. *And indeed,* $1 - \text{BinomialDist}(9; 100, .02818) = .00054$. *But now here is the really interesting part. Suppose that the probability of default was just a little higher,* .06. *This is a 20% increase in the risk of the underlying mortgages. What happens to the default probability of these tranches? We calculate* $1 - \text{BinomialDist}(9; 100, .06) = .0775$. *Notice that the percentage increase in risk of this tranche is huge:* $\frac{.0775-.0281}{.0281} = \%175$. *For the CDO it is even bigger. We calculate* $1 - \text{BinomialDist}(9; 100, .0775) = 0.2467$, *which is represents a* $\frac{.2467-.00054}{.00054} = 45339\%$ *increase in risk. That is quite a significant change, so we can conclude that the CDO tranches are indeed quite sensitive to the underlying default probabilities.*