

## Endogenous Growth Models

Barry W. Ickes  
 Department of Economics  
 Penn State University  
 University Park, PA 16802

### Introduction

Much of the recent literature distinguishes between exogenous and endogenous growth models. We have studied the former, and now we look at the latter. What is the difference? The importance difference is that in the former the steady-state growth rate is determined exogenously, e.g., technical change. In the latter, it is determined endogenously. The models are interesting because they often leave a role for policy.

One of the main reasons why economists have grown interested in endogenous growth is because of an empirical puzzle. The neo-classical model predicts that countries with low per-capita incomes grow faster than those with high  $y$ , so that over time per-capita incomes converge. At first the data we had seemed to support this prediction, but soon it became evident that this result was a product of sample selection; the early data sets included only those countries that had industrialized, so their per-capita incomes had been growing closer over time.<sup>1</sup> When attention was shifted to broader data sets it became apparent that poor countries were not converging, on average. For every South Korea there was a Phillipines, where per-capita income over 1960-85 grew at a slightly *lower* rate than in the US despite the fact that in 1960  $y_p = 0.1y_{US}$ .

This observation presents a problem for the standard model of growth. To see why assume that output takes the simple Cobb-Douglas form,  $Y = A(t)K^{1-\beta}L^\beta$ . In this expression  $A(t)$  denotes the level of technology, and its dependence on time denotes the exogenous rate of technical change. Let  $s$  be the constant rate of savings in the economy. We can write the expression for output in per-capita terms:

$$\frac{Y}{L} \equiv y = A(t)k^{1-\beta} \quad (1)$$

If we take logs of (1) and differentiate with respect to time, we get the familiar growth accounting equation:

$$\begin{aligned} \frac{\dot{y}}{y} &= \frac{\dot{A}}{A} + (1 - \beta)\frac{\dot{k}}{k} \\ \text{or} \\ \hat{y} &= (1 - \beta)\hat{k} + \hat{A} \end{aligned} \quad (2)$$

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<sup>1</sup>Romer (1994) provides an insightful analysis of how empirical observation motivated the development of endogenous growth models.

Equation (2) is the familiar growth accounting equation which relates growth in per-capita income to growth in the capital labor ratio (intensive growth) and growth in productivity. The coefficient  $\beta$  is labor's share of GNP, so it can be taken from data. Since we can measure  $y$  and  $k$ , (2) can be used to measure productivity growth.

Now we can derive an expression for  $\hat{k}$  by noting that  $\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - n$ , where  $n$  is the growth rate of the

labor force. Since  $\dot{K} = sY$ , we can write  $\hat{k}$  as:

$$\begin{aligned}\hat{k} &= K^{-1}(sA(t)K^{1-\beta}L^\beta) - n \\ &= K^{-1}(sLA(t)k^{1-\beta}) - n \\ &= sk^{-1}A(t)k^{1-\beta} - n = sk^{-1}y - n\end{aligned}\tag{3}$$

But we know from (1) that  $k^{-1} = A(t)^{\frac{1}{1-\beta}}y^{\frac{-1}{1-\beta}}$ , so that we can write  $\hat{k}$  as:

$$\begin{aligned}\hat{k} &= syA(t)^{\frac{1}{1-\beta}}y^{\frac{-1}{1-\beta}} - n \\ &= sA(t)^{\frac{1}{1-\beta}}y^{\frac{-\beta}{1-\beta}} - n\end{aligned}\tag{4}$$

and thus we can re-write (2) as:

$$\hat{y} = (1 - \beta)\left[sA(t)^{\frac{1}{1-\beta}}y^{\frac{-\beta}{1-\beta}} - n\right] + \hat{A}.\tag{5}$$

From (5) we can see how, outside the steady state, variation in the investment rate and in  $y$  should translate into variation in the growth rate. For a broad sample of countries the value of  $\beta$  is around 0.6. Recall that this is labor's share in income, given competition. This means that the exponent on  $y$  in (5) is -1.5. Now we perform the following experiment. We take a country, like the Phillipines, that had  $y$  in 1960 about 0.1 that of the US. Since  $0.1^{-1.5}$  is about 30, equation (5) says that the US would have required a savings rate 30 times as large as the Phillipines to grow at the same rate! If we used  $\beta = 2/3$  rather than 0.6, the required

difference would be 100 times. The evidence clearly shows that US savings was not nearly this high relative to Phillipines.

Of course this simple calculation assumes that the level of technology  $A(t)$  is the same in the two countries. Given the same  $A$ , the only way to account for the difference in  $y$  between the countries is in the size of the capital stock. Filipino workers must be using less capital than their American counterparts, accounting for the lower per-capita incomes. From (1) it is clear that the ratio of  $k_p$  to  $k_{US}$  is  $0.1^{1/(1-\beta)}$ , which is on the order of 0.3 percent. This implies that the marginal product of capital is much higher in the Phillipines than in the US, so a correspondingly higher investment rate is needed in the latter.

The problem with this analysis is that the savings rate in the US is at most twice that of the Phillipines, not 30 or 100 times larger.

To reconcile the data with the theory it is critical to somehow reduce  $\beta$ , so that labor is relatively less important in production. In that case diminishing returns to capital accumulation will set in much slower. The problem is to explain why the share of labor in national income is so much larger than  $\beta$ , or in other words, why labor is paid so much more than its marginal product while capital is paid so much less. Endogenous growth theories have developed to explain this.<sup>2</sup>

From a technical point of view, one can easily see the difference between exogenous and endogenous growth models. It is convenient to begin with the assumption of a fixed savings rate (i.e., no optimizing). Assume that output is Cobb-Douglas:

$$Y = K_t^\beta L_t^\alpha \quad (6)$$

Net investment,  $dK/dt$ , is savings minus depreciation, so:

$$\dot{K}_t = sK_t^\beta L_t^\alpha - \delta K_t \quad (7)$$

where  $s$  is the constant rate of savings. Now we are interested in  $\dot{k}/k$ . Using the definition of  $k/k$  and (6):

$$\begin{aligned} \frac{\dot{k}}{k} &= \frac{\dot{K}}{K} - \frac{\dot{L}}{L} = \frac{sK_t^\beta L_t^\alpha - \delta K_t}{K_t} - n \\ &= sK_t^{\beta-1} L_t^\alpha - \delta - n \end{aligned} \quad (8)$$

Now if we multiply through by  $k$  we obtain:

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<sup>2</sup>Although they are also motivated other puzzles, such as the question of why people from poor countries migrate to rich countries? Why would human capital move from where it is scarce to where it is abundant?

$$\begin{aligned}
\dot{k} &= sK^\beta L^{\alpha-1} - (n + \delta)k \\
&= \frac{L^\beta}{L^\beta} (sK^\beta L^{\alpha-1}) - (n + \delta)k \\
&= sk^\beta L^{\beta+\alpha-1} - (n + \delta)k
\end{aligned} \tag{9}$$

Our interest is in steady states, so we need an expression for  $\dot{k}/k$ , so divide both sides of (4) by  $k$ :

$$\frac{\dot{k}}{k} = sk^{\beta-1} L^{\beta+\alpha-1} - (n + \delta) \tag{10}$$

If we take logs of both sides and differentiate with respect to time:

$$\begin{aligned}
0 &= (\beta - 1) \frac{\dot{k}}{k} + n(\beta + \alpha - 1) \\
\Rightarrow \\
0 &= (\beta - 1) \gamma_k + n(\beta + \alpha - 1)
\end{aligned} \tag{11}$$

where  $\gamma_k \equiv \dot{k}/k$  and we have used the fact that  $n$ ,  $\delta$ , and  $s$  are constant in the steady state, so that their time derivatives are zero. Similarly, we have used the fact that in a steady state  $\dot{k}/k$  is constant, hence the time derivative of the LHS of (11) is zero.

Now if there are constant returns to scale in capital and labor (as in the standard model), then  $\beta + \alpha = 1$ , and the last term in (11) is zero. *This implies that the only steady state consistent with the model is one with zero growth.* This follows since  $\beta < 1$ . Notice that if there were constant returns to capital accumulation (i.e.,  $\beta = 1$ ), then there could be steady states with  $\gamma_k \neq 0$ .

Of course this result has nothing to do with a fixed savings rate. In a growth model with optimizing individuals the time path of consumption will be constant if the rate of interest is equal to the rate of time preference. In a representative agent model individual and aggregate consumption coincide, so there is no growth in this case. If, on the other hand, the rate of interest exceeded the rate of time preference, there is an incentive for agents to increase consumption in the future, and thus the time path of consumption is upward sloping. Now the standard arbitrage argument suggests that the rate of interest is equal to the marginal product of capital. Hence, if the technology is such that the marginal product of capital goes to zero as capital per worker increases, it follows that the rate of interest will eventually equal the rate of time preference. At this point desired consumption is constant over time and the process of capital accumulation stops.

Jones and Manuelli (1990) point out that for growth to be possible the marginal product of capital must be bounded from below. That is, as  $k$  goes to infinity,  $f(k)$  goes to some lower bound,  $B$ . If this is the case, and if  $B >$  than the rate of time preference, then continuous growth is feasible.

To make this clearer, note that from (10) we can write the growth rate of capital,  $\gamma_k$ , as:

$$\gamma_k = \frac{sf(k)}{k} - (n + \delta). \quad (12)$$

For  $\gamma_k > 0$  in steady state, it must be the case that  $\frac{f(k)}{k} > \frac{n + \delta}{s}$  as  $k \rightarrow \infty$ . This implies, in turn, that

$\lim_{k \rightarrow \infty} \left[ \frac{f(k)}{k} \right] > \frac{n + \delta}{s}$  is necessary and sufficient for a steady state with positive growth. By l'Hopital's rule:

$$\lim_{k \rightarrow \infty} \left[ \frac{f(k)}{k} \right] = \lim_{k \rightarrow \infty} [f'(k)] > \frac{n + \delta}{s} > 0. \quad (15)$$

where the inequality results from the condition for  $\gamma_k$  to be positive. But (13) clearly violates the Inada conditions, since  $f'(k)$  goes to zero as  $k$  goes to infinity. Thus standard production functions are inconsistent with endogenous growth.

To remedy this situation, Jones and Manuelli suggest we consider production functions of the type

$$\begin{aligned} Y &= AK^\alpha L^{1-\alpha} + bK \\ &\text{or} \\ y &= Ak^\alpha + bk \end{aligned} \quad (16)$$

which implies that

$$\frac{f(k)}{k} = Ak^{-(1-\alpha)} + b$$

so that (13) is satisfied. Notice that with the Jones-Manuelli bound, as  $k$  goes to infinity  $f_k$  goes to  $b > 0$ . Production functions that resemble (14) have display diminishing returns to capital, *up to a point*. Thus such an economy will display transition dynamics, and it will display positive growth in the steady state.<sup>3</sup>

Notice that convex technologies can be consistent with such a bound. A convex technology requires that  $f(k)$  be a decreasing function of  $k$ , but not that it decrease without bound. A simple example would be a production function of the form:  $F(K, L) = AK^\alpha L^{1-\alpha} + bK$ . Of course this example is not all that appealing since it implies that labor's share of national income goes to zero.

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<sup>3</sup>As long as  $b \geq \frac{n + \delta}{s}$ .

The essence of endogenous growth models is to somehow implement a Jones-Manuelli bound on the marginal product of capital. In essence, what the endogenous growth models do is impose constant returns on the *reproducible factors of production* (i.e.  $\beta = 1$ ). This kind of model gives no role to *non-reproducible factors of production*, such as land and labor, and gives primary focus to capital. We shall examine this model (Rebelo's model) shortly. Note that the model is not ignoring labor *per se*, but labor devoid of human capital. The implied production function is

$$Y_t = AK_t \quad (17)$$

The idea here is that the labor force improves in quality with human capital accumulation, and that savings is devoted to both. Human and physical capital are combined together in a broad measure, and a production function like (15) above results. This is the Lucas-Uzawa approach.

An alternative possibility is to assume that there exist increasing returns to scale. If  $n = 0$ , we can have non-reproducible inputs ( $\alpha > 0$ ), and steady state growth,  $\gamma_k > 0$ , if there are *constant returns to the inputs that can be accumulated* ( $\beta = 1$ ). But this means that there are increasing returns to scale  $\alpha + \beta > 1$ .

Notice that with increasing returns to scale we have some additional problems that arise because we cannot have competitive prices. There are two main ways to get around this problem. The first (originally due to Marshall, and is now associated with Romer via Arrow) is to assume IRS at the aggregate level, but to assume CRS at the firm level. The idea is that there are spillovers that are external to the firm, but that none of the firms take them into account. Hence all the firms face "concave" problems, but the economy as a whole faces an IRS production function which can generate endogenous growth. It is immediately apparent from this description that the equilibria in such a model will not be efficient.

The (Cobb Douglas) production function is

$$Y_t = AK_t^\beta L_t^{1-\beta} \kappa_t^\Psi \quad (18)$$

where  $K_t$  is private capital and  $\kappa_t$  is aggregate capital in the economy. Individuals firms assume that they cannot affect the aggregate stock of capital, so they take  $\kappa$  as given. This makes the firm's problem quite standard. But in the aggregate,  $\sum_i K_{ti} = \kappa_t$ . Thus the aggregate production function will be

$$Y_t = AK_t^{\beta+\Psi} L_t^{1-\beta} \quad (19)$$

Now examine (11) in the context of (13). Since (11) is derived from the aggregate production function, we replace  $\beta$  in (11) with  $\beta' \equiv \beta + \Psi$ . Now if  $\beta' = 1$ , then we can have steady state growth. Hence we have *constant returns to capital in an increasing returns to scale world*. The reason is that the spillover is *external* to the firm. Modelling externalities in this way we get around the problem of in-existence of competitive equilibrium. But the equilibria will be *non-optimal*. In the Romer model these externalities take the form of knowledge spillovers, as we shall see.

An alternative way to get around the problem of existence of competitive equilibrium with IRS is to drop the assumption of competitive behavior. With imperfect competition factor returns do not exhaust total output. Hence there are rents that can be assigned to activities that are not directly productive, but may contribute to the expansion of the frontiers of knowledge such as research and development. Many economists believe that this is an important source of economic growth.

There have also been models that include both externalities *and* imperfect competition (e.g., Grossman and Helpman). In these models firms undertake R&D in order to introduce new goods (a product differentiation motive), but this activity also increases the general stock of knowledge. This, in turn, makes it less costly to undertake further research (which insures that it will continue) and it increases the productivity of other inputs. Since the stock of knowledge grows at a constant rate, so does output.

## Endogenous Growth Models

Before looking at some specific models in more detail it is worthwhile to look again at the distinction between endogenous and exogenous growth models. We have seen that the key to the former was the inexistence of diminishing returns to the inputs that can be accumulated. Hence the return to investment in all these models end up being a constant,  $A^*$ :

$$r = A^* \tag{1}$$

Let us further suppose that consumption is determined according to an intertemporal optimization problem, so that the MGR results. We will derive this again shortly (see equation (18) below). Since our concern is with endogenous growth models, rather than let the growth rate be  $n$ , we denote it by  $\gamma_k$ , which is constant in the steady state:<sup>4</sup>

$$r = \rho + \gamma_k \tag{2}$$

We have two equations in  $r$  and  $\gamma_k$ , so we can plot this as in figure 1. Notice that the intersection of the two curves yields the equilibrium growth rate. Suppose that  $A^*$  increases. Then it is apparent from the figure that  $\gamma_k$  will increase. Hence the focus of attention in endogenous growth models is to understand the determinants of  $A^*$ . In particular, concern centers on the role of policy in affecting  $A^*$ .<sup>5</sup>

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<sup>4</sup>Notice from (8), below, we have  $r = \rho + \sigma\gamma$ . Hence let  $\sigma = 1$ , and we have expression (2).

<sup>5</sup>Less attention is devoted to the determinants of  $\rho$ , or of the inter-temporal elasticity of consumption (the inverse of which multiplies  $\gamma_k$ , but which I have assumed to be unity). One might consider, however, that some countries are more willing to defer consumption than others. This has been relatively unstudied, however.

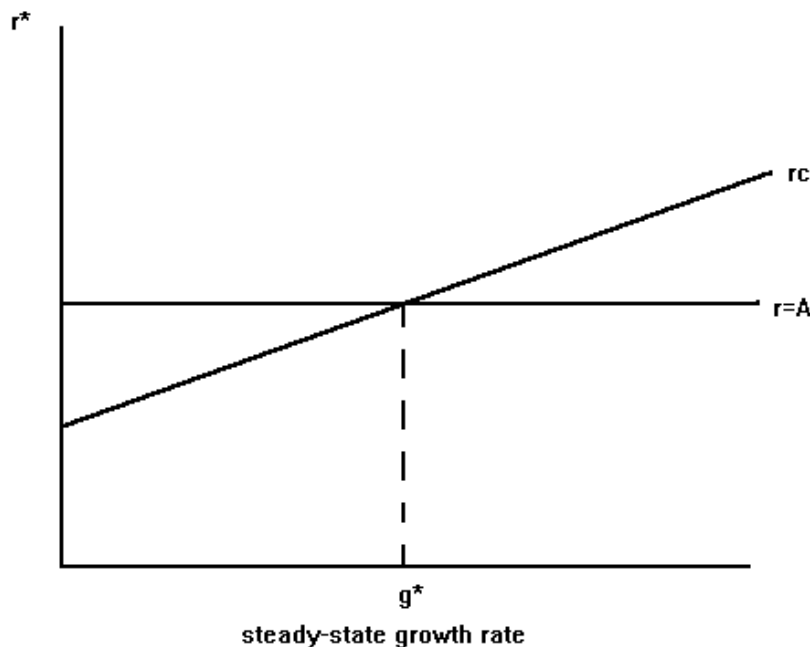


Figure 1

Now we contrast this with the exogenous growth model. In this case we still have equation (2), but now the growth rate is exogenous. Hence we have figure 2. Comparing the two pictures we see that in the exogenous growth model, a change in any of the parameters that determine the return to consumption affect  $r^*$ , but not the growth rate. In the endogenous growth model such changes affect the growth rate, but not the interest rate.

### Rebelo's Model

The simplest introduction into this literature is Rebelo's model. Rebelo assumes that the production function is linear in the only input, capital. Hence there are constant returns to scale and constant returns to capital. The production function is:

$$Y = F(K, L) = AK \quad (3)$$

where  $A$  is an exogenous constant, and  $K$  is aggregate capital broadly defined. Thus  $K$  can include not just physical capital but also human capital as well as the stock of knowledge and even financial capital.



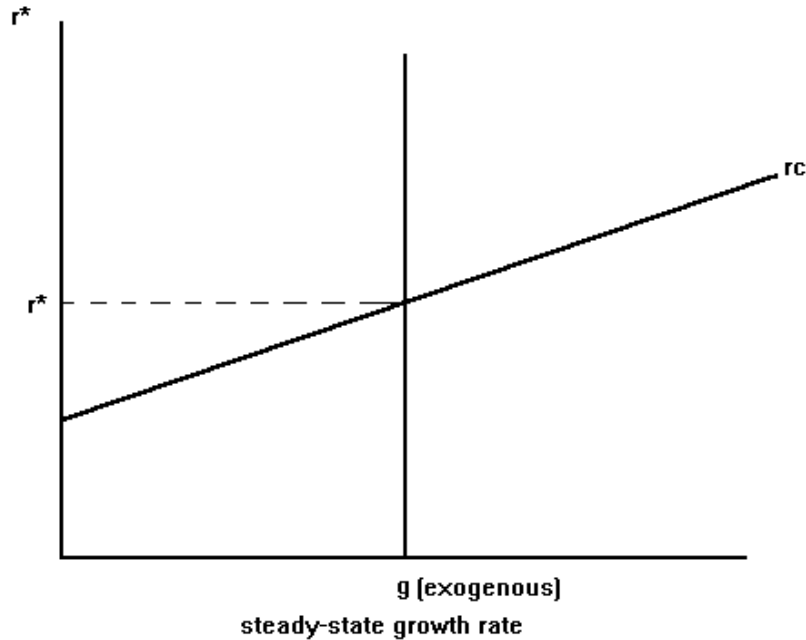


Figure 2

For simplicity we assume that  $n = \delta = 0$ . The utility function that households maximize is of the constant intertemporal elasticity of substitution form:

$$U(0) = \int_0^{\infty} e^{-\rho t} \left( \frac{c_t^{1-\sigma} - 1}{1-\sigma} \right) dt \quad (4)$$

where  $\rho$  is the rate of time preference and  $\sigma^{-1}$  is the intertemporal elasticity of substitution. Let  $b$  be the financial wealth of households. Financial wealth is physical capital plus bonds. In the aggregate the net supply of bonds equals zero, so  $b = k$ . Households thus face a financial constraint of the form:

$$\dot{b}_t = r b_t - c_t \quad (5)$$

Agents maximize (4) subject to (5) and appropriate boundary conditions. The Hamiltonian for the problem is:

$$H = e^{-\rho t} \left[ \frac{c_t^{1-\sigma} - 1}{1-\sigma} \right] + v(Ab_t - c_t) \quad (6)$$

where  $v$  is the multiplier on the costate variable, and I have used  $A = r$ .<sup>6</sup> The first-order conditions for this problem include:

$$\begin{aligned} H_c &= e^{-\rho t} c_t^{-\sigma} - v_t = 0 & (i) \\ H_b &= -\dot{v} \Leftrightarrow \dot{v} = -vA & (ii) \end{aligned} \quad (7)$$

and a transversality condition.

Now take logs of (i) and differentiate with respect to time to get

$$\frac{\dot{v}}{v} = -\rho - \sigma \frac{\dot{c}}{c} = -\rho - \sigma \gamma \quad (8)$$

where  $\gamma$  is the balanced growth rate of consumption (and capital). Note that (ii) implies that  $\dot{v}/v = -A$ , so that from (8) it follows that  $-A = -\rho - \sigma\gamma$ . Hence

<sup>6</sup>Alternatively, we could write the present value Hamiltonian as:

$$H_t = e^{-\rho t} \left\{ \frac{c_t^{-\sigma} - 1}{1-\sigma} + q_t(Ab_t - c_t) \right\}$$

where  $q_t = ve^{-\rho t}$ , is the current value of the costate variable. The F.O.C. are then

$$H_c = c_t^{-\sigma} - q_t = 0$$

and

$$-H_b = \left( \frac{d}{dt} \right) q e^{-\rho t}$$

Note that  $-H_b$  is equal to  $-q_t A e^{-\rho t}$ , so that the second condition can be written as:

$$-q_t A e^{-\rho t} = \rho q_t e^{-\rho t} + \dot{q}_t e^{-\rho t}$$

$$\dot{q} = -q_t(A - \rho) \Rightarrow \frac{\dot{q}}{q} = -(A - \rho)$$

Now we can substitute for  $q_t$  from the first condition, and we obtain:

$$-(A - \rho) = \frac{-\sigma \dot{c} c^{-\sigma-1}}{c^{-\sigma}} = \frac{-\sigma \dot{c}}{c}$$

hence,

$$\frac{\dot{c}}{c} = \frac{A - \rho}{\sigma}$$

which is equation (9) in the text.

$$\frac{\dot{c}}{c} = \gamma = \frac{A - \rho}{\sigma} \quad (9)$$

Note that (9) implies that  $A = \sigma\gamma + \rho$ , which is the MGR (recall that  $f(k_t) = A$  in this model). This is just the curve  $rc$  in figure 1.

What about  $k$ ? If we substitute  $\dot{k} = b$  into (5) we have  $\dot{k} = Ak - c$ . Divide both sides by  $k$ , and note that in a steady state  $\dot{k}/k$  is constant. Then if we take logs of both sides and differentiate with respect to time we obtain  $\dot{c}/c = \dot{k}/k$ . Note from (9), however, that this balanced growth rate need not be zero. As long as  $A$  is large enough, we will have positive steady state growth.

Now let us look at the interaction of the savings rate and the rate of growth. We have:

$$\frac{s}{y} = \frac{\dot{k}}{y} = \left(\frac{\dot{k}}{k}\right)\left(\frac{k}{y}\right) = \gamma\left(\frac{1}{A}\right) = \frac{1 - \rho}{\sigma A} \quad (10)$$

Hence the growth rate of an economy depends on its savings rate and on the productivity of its technology, since (10) implies that  $\gamma = (s/y)A$ . Furthermore, the last equality in (10) implies that  $s/y$  depends on  $\rho$  and  $\sigma$ .<sup>7</sup> If  $\rho$  is small, society is more patient, and savings and the growth rate will be higher. Similarly if agents are more willing to substitute consumption intertemporally,  $\sigma$  low, then again savings and growth will be higher. What remains to be explained is the determination of  $A$ , which we shall discuss in the context of the Romer and Lucas models.

Note also that this model does *not* predict convergence. To see this assume that countries have the same parameters,  $(A, \sigma, \rho)$ , but that they start with different initial capital stocks. Since they grow at the same rates (by virtue of (10)), their levels of *per-capita* income cannot converge. What is perhaps of more interest, is that given identical preferences, different values of  $A$  imply different growth rates, so that if poor countries have low  $A$ , they will not catch up. This seems to imply a role for policy in economic development, if it can affect  $A$ . We now must turn to its determination.

## Romer's Model

Romer started the endogenous growth literature by considering a model with increasing returns to scale at the economy-wide level, but constant returns to scale at the firm level. The model then supports a competitive equilibrium, but this equilibrium is non-optimal. A higher growth rate could be achieved if the

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<sup>7</sup>And, of course, on  $A$  which determines the rate of return.

externality associated with investment could be internalized. This alone made the model popular, and it has spawned a large literature.

Romer follows Arrow's seminal work on the economics of learning by doing. Arrow noted from case studies that there was strong evidence that experience and increasing productivity were associated. He argued that a good measure of increase in experience is investment, because "each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning takes place with continuous new stimuli" (157). Arrow then indexes experience by cumulative investment.

Let the production function for firm  $i$  be:

$$Y_i = F(K_i, A(t)L_i) \quad (11)$$

where  $A(t)$  reflects the stock of knowledge at time  $t$ . The idea is that labor is more productive given the accumulation of knowledge. This, in turn, depends on experience which is a function of past investment of *all* firms in the economy. Hence:

$$G(t) = \int_{-\infty}^t I(v)dv = \kappa(t) \quad (12)$$

with no depreciation, the sum of past investment is equal to the aggregate capital stock. The learning by doing assumption is that  $A(t) = G(t)^\eta$ , with  $\eta < 1$ . This means that investment raises the productivity of labor, but at a decreasing rate. Hence we can rewrite (11) as:

$$Y_i = F(K_i, L_i, \kappa) = K_i^\beta L_i^{(1-\beta)} \kappa^\eta \quad (13)$$

Notice that (13) is constant returns to scale holding  $\kappa$  fixed, but that it is increasing returns to scale when we consider the three "inputs" at the same time. With a large number of firms we can assume that firms take  $\kappa$  as given in their maximization problem. This will be the source of the externality. A command planner would consider the effect of investment on production via the experience gained. A firm will not.

If we aggregate across firms we can write the aggregate production function as:

$$Y = F(K, L, \kappa) = K^\beta L^{(1-\beta)} \kappa^\eta \quad (14)$$

Dividing through by  $L$  gives us a per-capita production function:

$$y = k^\beta \kappa^\eta \quad (15)$$

where  $k = K/L$ , and  $y = Y/L$ . Assume that the households maximize a utility function as in (4), subject to a dynamic constraint (again ignoring population growth):

$$\dot{k} = k^\beta \kappa^\eta - c \quad (16)$$

To obtain the conditions of the competitive equilibrium we set up the household's decision, as before, but remembering that the household chooses  $k$  assuming that  $\kappa$  is given. The Hamiltonian for this problem is

$$H = e^{-\rho t} \left[ \frac{c_t^{1-\sigma} - 1}{1-\sigma} \right] + v(k_t^\beta \kappa_t^\eta - c_t) \quad (17)$$

and the first order conditions will include:

$$\begin{aligned} H_c &= e^{-\rho t} c_t^{-\sigma} - v_t = 0 & (i) \\ H_k &= -\dot{v} \Leftrightarrow \dot{v} = -v(\beta k_t^{-(1-\beta)} \kappa_t^\eta) & (ii) \end{aligned} \quad (18)$$

Equilibrium in the capital market requires that  $\kappa = Lk$ . Now if we take logs of (18i) and differentiate with respect to time we get

$$\frac{\dot{v}}{v} = -\rho - \sigma \frac{\dot{c}}{c} \quad \Leftrightarrow \quad \frac{\dot{c}}{c} = -\sigma^{-1} \left( \frac{\dot{v}}{v} + \rho \right) \quad (19)$$

Now substitute for  $\dot{v}/v$  from (27ii), using  $\kappa = Lk$ , and we obtain:

$$\frac{\dot{c}}{c} = \sigma^{-1} (L^\eta \beta k^{-(1-\beta-\eta)} - \rho) \equiv \gamma \quad (20)$$

Expression (20) relates the growth of consumption to the difference between the marginal product of capital and the discount rate, times the factor of proportionality,  $\sigma^{-1}$ . Now divide both sides of (16) by  $k$ , take logs and time derivatives, and we can show that  $\dot{k}/k = \gamma$ . Note that this model has the counterfactual implication that the growth rate of consumption is increasing in the population,  $L$ . This is due to the nature of the scale effect. If we had related experience to the *average* capital stock instead, then this would not occur.

How do we interpret these results? Let  $\beta L^\eta = A^*$ . Then if  $\beta + \eta = 1$ , (20) reduces to

$$\gamma = \sigma^{-1} (A^* - \rho) \quad (21)$$

which is isomorphic to Rebelo's model (equation (9)), except that in this case the social and private marginal products of capital are unequal. Thus Romer's model also generates endogenous growth, when  $\beta + \eta = 1$ .

This would not be true, however, if  $\beta + \eta < 1$ . In this case the model is identical to the standard exogenous growth model. In the context of (6), we would have  $\hat{\beta} = \beta + \eta$  in place of  $\beta$  as capital's share in (6). But then the only steady state would be the one with  $\gamma = 0$ . This is an important point. It means that increasing returns are not a *sufficient* condition for endogenous growth. What we need is sufficiently large increasing returns so that  $\beta + \eta = 1$ .

Now let us consider what the growth rate would be if a planner were to choose levels of investment. This is important since we know that private agents are not considering the spillover that arises from

investment. Recall that  $\kappa = Lk$ . Hence we could rewrite the last term in the Hamiltonian as  $v(k^{\beta+\eta}L^\eta - c)$ . Consequently we can rewrite (20) as:

$$\gamma_{planner} = \sigma^{-1}(L^\eta(\beta + \eta)k^{-(1-\beta-\eta)} - \rho) \tag{23}$$

since  $\eta > 0$ , this is clearly greater than the growth rate in the competitive equilibrium. This is clearly due to the failure of private agents to take into account the effects of their investment on aggregate capital, and hence on learning by doing. Private agents fail to internalize the spillover in production; they under-invest, and, therefore, they "undergrow."

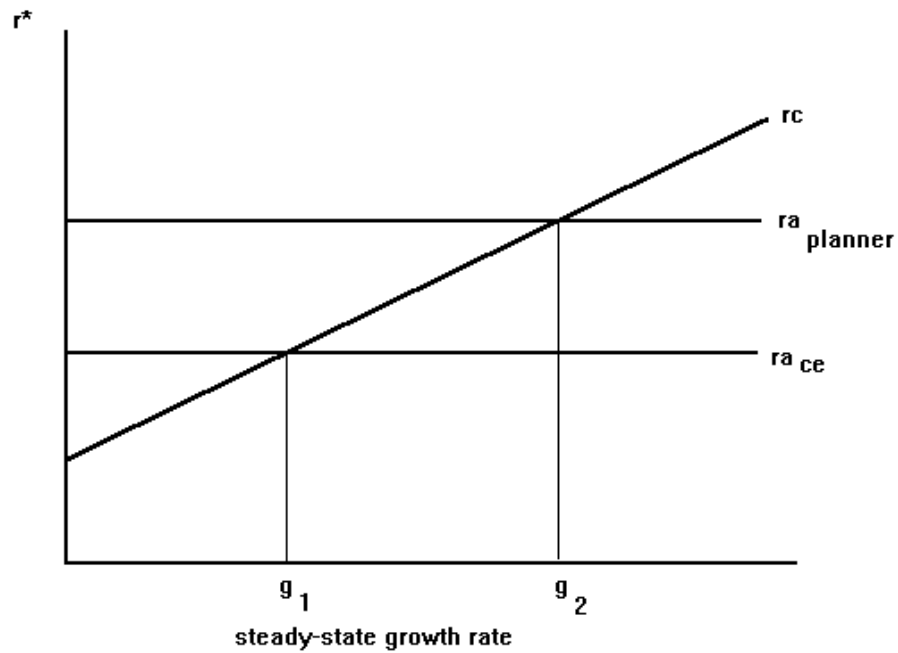


Figure 3

It is instructive to put this in terms of a figure similar to 1. First note that in competitive equilibrium firms invest until the marginal product of capital is equal to  $r$ . Since the production function is

$$y = k^\beta k^\eta L^\eta \tag{24}$$

the *private* marginal product is  $\beta k^{\beta-1} k^\eta L^\eta = \beta k^{\beta+\eta-1} L^\eta$ , due to the failure of private agents to consider the effect of investment on  $\kappa$ . But for the planner the relevant marginal product is

$$r = (\beta + \eta)k^{\beta+\eta-1} L^\eta \tag{25}$$

which is clearly larger if  $\eta > 0$ . Hence in terms of figure 3, the  $r^* = A$  line for the planner ( $r_{i,planner}$ ) lies above that for the competitive equilibrium. But this clearly gives a lower equilibrium rate of growth in the latter case.

But it is not just the fact that the competitive equilibrium and the solution to the planners problem differ that is interesting. The model also allows one to introduce significant effects from policy. Suppose, for example, that tax laws are changed that increase the incentive to invest (an ITC for example). In the standard approach this may have some effect on the efficiency of the allocation of resources, and hence effect the *level* of income today, but it will leave the growth rate unaffected. In this model, on the other hand, an increase in investment incentives will shift  $r_i$  upwards, and hence raise  $\gamma$ . This is clearly optimal as long as the new  $r_{i_{ce}}$  does not lie above  $r_{i_{planner}}$  in figure 3.

There is still more to the point, however, if we think about a case where the effect of a policy is to raise the growth rate and lower the level of output. Now if we consider a long time horizon, growth rate effects will dominate level effects. Romer gives the following numerical example, which stems from Jorgenson's evaluation of the effects of the 1986 tax reform.

Suppose that elimination (say by the 1986 tax reform) of ITC and uniform taxing of capital gains and dividend income reduces distortions and raises the level of output by 1% (forever). Suppose that interest rates were 5% and that the growth rate of output was 3%. The present value of future GNP is  $1.0/(0.05 - 0.03) = 1.0/0.02 = 50$  times current GNP. Now suppose that the level of GNP rises by 1%, and that growth is *exogenous* (and hence unchanged). The present value of future GNP rises to  $1.01/.02 = 50.5$  times pre-reform GNP. So the effect of the tax reform is to increase wealth by half a year's GNP. Now suppose that the tax reform also effects the growth rate, reducing it from 3% to 2.9%. The effect of the policy then is to make the present value of future GNP  $1.01/(.05 - .029) = 1.01/.021 = 48.1$  times current GNP. *So output falls by 2 years worth of output*. It is not surprising that growth rate effects dominate level effects, and that is the main point of the example. It is instructive to see that the assumption that the growth rate will be unaffected *can* have serious implications for the evaluation of policy.

Many people relate endogenous growth to increasing returns to scale. This is, to a large extent, due to the fact that Romer's model really got this literature going.<sup>8</sup> But it is clear from even our brief survey, that increasing returns are neither necessary, nor sufficient to generate endogenous growth. The former statement follows from Rebelo's model which generates endogenous growth without increasing returns. The latter statement follows from Romer's model, since we have seen that for endogenous growth to be possible, we need sufficiently large externalities.

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<sup>8</sup>Although Romer recognized this point, many still took increasing returns to be the central lesson of the model.

## A Simple Human Capital Model

Let us begin by examining a simple human capital model. We really should examine a two-sector model, but instead we will assume that output can be used for human and physical capital. There is constant returns to scale for both inputs. Our assumptions about production and output can be written as:

$$Y = AK^\alpha H^{1-\alpha} = C + I_K + I_H \quad (1)$$

with the following accumulation conditions:

$$\begin{aligned} \dot{K} &= I_K - \delta K \\ \dot{H} &= I_H - \delta H \end{aligned} \quad (2)$$

Notice that in (2) we have assumed that the rate of depreciation of the human and physical capital stock are equal. This is not essential, but it radically simplifies the following.

We can write the present value Hamiltonian as:

$$\begin{aligned} J &= u(c)e^{-\rho t} + v(I_K - \delta K) + \mu(I_H - \delta H) + \omega(AK^\alpha H^{1-\alpha} - C - I_K - I_H) \\ &\text{or} \\ J &= u(c)e^{-\rho t} + \lambda(AK^\alpha H^{1-\alpha} - C - \delta K - \delta I_H) + \mu(I_H - \delta H) \end{aligned} \quad (3)$$

If we use standard CARRA utility,  $u(c) = \frac{c^{1-\theta} - 1}{1-\theta}$ , and appropriate nonnegativity conditions,  $I_K, I_H \geq 0$

(which we will ignore for a bit), we obtain from the FONC the conditions for the growth of consumption:

$$\frac{\dot{c}}{c} \equiv \gamma_c = \frac{1}{\theta} \left[ A\alpha \left( \frac{K}{H} \right)^{-(1-\alpha)} - \delta - \rho \right] \quad (5)$$

where we note that the first two terms inside the brackets of (4) is the net marginal product of physical capital.

Since agents can invest in physical and human capital, and since the cost (in terms of output is the same), it follows that the net marginal product of physical capital should equal the net marginal product of human capital. The net marginal product of human capital is  $MP_H = A(1-\alpha)(K/H)^\alpha - \delta$ . Thus, setting the two equal yields:



$$A\alpha\left(\frac{K}{H}\right)^{-(1-\alpha)} - \delta = A(1-\alpha)\left(\frac{K}{H}\right)^{\alpha} - \delta \quad (7)$$

which implies that:

$$\frac{K}{H} = \frac{\alpha}{1-\alpha} \quad (8)$$

Given the structure of the model (6) makes perfect sense. Households can save in either physical or human forms, and the proportions in which they do this must be related to the relative productivity of the two activities. Notice that (6) also implies that in the steady state the ratio of physical to human capital will be constant.

Using (6) in the expression for the  $MP_K$  we can write the rate of return to investment,  $r^*$ , as:

$$r^* = A\alpha^{\alpha}(1-\alpha)^{1-\alpha} - \delta \quad (9)$$

Clearly  $r^*$  is constant because of the constant returns to scale with respect to physical and human capital.

Note further that if  $K/H$  is constant then  $\gamma_c$ , the steady-state growth rate of consumption, is constant and equal to:

$$\gamma^* = \frac{1}{\theta} [A\alpha^{\alpha}(1-\alpha)^{1-\alpha} - \delta - \rho]. \quad (10)$$

It is interesting to note that if we substitute from (6) into the production function (1) we obtain an expression that is clearly of the AK type:

$$Y = AK\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} \quad (11)$$

From (9) it is clear that for any given value of  $\alpha$  this is an AK model.

What happens if the initial ratio of physical to human capital,  $\frac{K(0)}{H(0)}$ , differs from  $\frac{\alpha}{1-\alpha}$ ?

Instantaneous adjustment makes no sense. An economy cannot turn K into H overnight. This would require an infinite rate of investment, and if the initial physical capital stock was too high, an infinitely negative rate of investment, which is clearly nonsense.

Suppose that human capital is initially too abundant, i.e.,

$$\frac{K(0)}{H(0)} < \frac{\alpha}{1-\alpha}$$

then households wish to reduce H relative to K, so  $I_H = 0$ . Human capital depreciates at rate  $\delta$ , by assumption, so

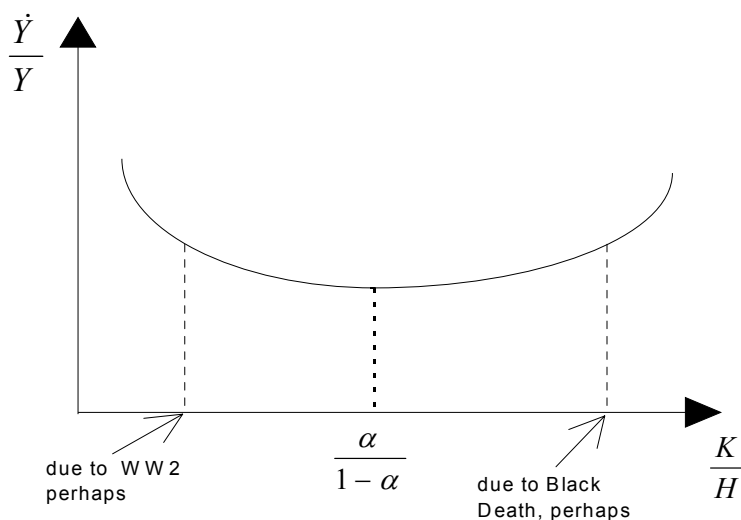
$\frac{\dot{H}}{H} = -\delta$ , or  $H(t) = H(0)e^{-\delta t}$ . This implies that

$$J = u(c)e^{-\rho t} + v(AK^\alpha H^{1-\alpha} - c - \delta K) \quad (16)$$

where  $I_H = 0$ . Notice that this is much like the Solow model, except that instead of population growth,  $n$ , we

have  $\frac{\dot{H}}{H} = -\delta$ . Now as  $\dot{K} > 0$ , eventually we have  $\frac{K}{H} \rightarrow \frac{\alpha}{1-\alpha}$ .

Once we reach the proper ratio of K to H, then investment in human capital can resume, and we return to the original solution with  $\gamma^* > 0$ . So the dynamics of the neoclassical model applies when H is abundant, and the K-H solution applies when we get to the steady state. In the transition,  $\frac{\dot{K}}{K}$  and  $\frac{\dot{Y}}{Y}$  decline monotonically over time. The marginal product of capital declines over time, but it is still greater than the marginal product of human capital until we reach  $\frac{K}{H} = \frac{\alpha}{1-\alpha}$ . The dynamics thus resemble:



As drawn in figure 4 adjustment is symmetric when the  $K/H$  ratio is wrong, but there is no reason why this must be so. One could add asymmetry; presumably it is harder to accumulate  $H$  than  $K$ . For both you need to save, but you need  $H$  to produce  $H$ . If so then the curve might be flatter to the right of the proper  $K/H$ . That is, when  $K$  is abundant the change in output is smaller than when  $H$  is abundant.

But to really deal with this analysis we need a two-sector model.

## Lucas's Human Capital Model

Lucas's paper on the "Mechanics of Economic Development" develops a model in which constant returns to scale in the inputs that can be accumulated is obtained by arguing that *all* inputs can be accumulated. Rather than rely on externalities, as in Romer, Lucas introduces human capital, rather than physical labor, in the production function. Agents invest in human capital through their "studies." All inputs of the production function can thus be accumulated. With a CRS production function, we have essentially Rebelo's model, where the broad measure of capital includes human *and* physical capital. Growth is then generated by assuming that the incentive to invest in human capital is nondecreasing in human capital. That is, Lucas postulates a production function of human capital which is constant returns to scale in human

capital. Hence the marginal product of human capital -- which determines the incentive to spend time studying -- is constant.

Let  $u$  be the fraction of non-leisure time agents spend working (i.e., producing the output good  $Y$ ), and let  $h$  be a measure of the average quality of workers and  $L$  be the number of bodies. Then  $uhL$  is to the total effective labor force. Population is going to grow at rate  $n$ , and is not important. Hence we let  $n = 1$  and write the production function in *per-capita terms*. We can write the production function as:

$$y = Ak^\beta [uh]^{1-\beta} \quad (1)$$

where the term  $uh$  is often called human capital. This production function clearly exhibits constant returns to scale in  $k$  and  $uh$ , since doubling these inputs doubles output. Note that if we interpret  $k^\beta [uh]^{(1-\beta)}$  as being a broad measure of capital, we are back to Rebelo's model (providing that the incentive to accumulate human capital does not decrease over time; otherwise we would cease accumulating it). Consequently this is sufficient to generate endogenous growth. We could stop here.

Lucas chooses to introduce, however, an externality in human capital to reflect the fact that people are more productive when they are around clever people (some people seem to be less productive in such situations, but that is another story). Let  $h_a$  be the average level of human capital in the labor force. Then we can write the production function as:

$$y = Ak^\beta [uh]^{1-\beta} h_a^\psi \quad (2)$$

where  $h_a^\psi$  represents the externality from average human capital. This externality is introduced, not to obtain endogenous growth (we have seen already that this is not needed), but to obtain some extra results on migration across countries.<sup>9</sup> In any event, agents choose to maximize the standard intertemporal utility function subject to the capital accumulation constraint:

$$\dot{k} = Ak^\beta [uh]^{1-\beta} h_a^\psi - c \quad (3)$$

To complete the model we need to specify how knowledge is accumulated. There are two ways to think about this. First, agents learn when they study! Thus we would relate human capital accumulation to

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<sup>9</sup>It is a bit strange to have the externality depend on *average human capital*. This seems to imply that if an agent with lower than average human capital migrates to the US everyone else becomes *less* productive!

time spend not working. Second, agents accumulate human capital through on the job training; this would relate to time working. For now we consider only the former approach:

$$\dot{h} = \phi h(1 - u) \quad (4)$$

Notice that (4) implies constant returns to scale in human capital accumulation, since  $\dot{h}/h$  is proportional to study time. This assumption is crucial. It is the driving force behind sustained growth in the model.

Let us now consider the agents problem. The representative agent chooses a stream of consumption *and* the time spend studying, taking  $h_a$  as given. The constraints are the asset accumulation equation (3) and the "study-time" equation (4). The Hamiltonian is

$$H() = e^{-\rho t} \left( \frac{c_t^{1-\sigma} - 1}{1-\sigma} \right) + v(Ak^\beta (uh)^{1-\beta} h_a^\psi - c) + \lambda[h\phi(1-u)] \quad (5)$$

The F.O.C. (with respect to  $C$ ,  $u$ ,  $k$ , and  $h$ , respectively) are:

$$\begin{aligned} e^{-\rho t} c^{-\sigma} &= v & (i) \\ v(Ak^\beta h^{1-\beta} (1-\beta)u^{-\beta} h_a^\psi) - \lambda h \phi &= 0 & (ii) \\ \dot{v} &= -v(\beta A k^{\beta-1} (uh)^{1-\beta} h_a^\psi) & (iii) \\ \dot{\lambda} &= -v((1-\beta)A k^\beta u^{1-\beta} h^{-\beta} h_a^\psi) - \lambda[\phi(1-u)] & (iv) \end{aligned} \quad (6)$$

Note, of course, that  $h_a = h$ , although agents ignore their own effect on it.

As before, take logs and derivatives of (i), and use (iii) and  $h_a = h$  to obtain:

$$\frac{\dot{c}}{c} \equiv \gamma = \sigma^{-1}(\beta A k^{-(1-\beta)} u^{1-\beta} h^{1+\psi-\beta} - \rho) \quad (7)$$

Now lets show that the growth rate of consumption is equal to the growth rate of capital. Divide both sides of (3) by  $k$  to get

$$\frac{\dot{k}}{k} \equiv \gamma_k = A k^{-(1-\beta)} u^{1-\beta} h^{1+\psi-\beta} - \frac{c}{k} \quad (8)$$

Comparing (8) with (7) it is apparent that the first part of the first term on the rhs of (8) is equal to  $(\gamma\sigma + \rho)/\beta$ . Hence if we take logs and derivative of both sides of (8), this term will drop out since all of its elements are

constants. Thus we will get  $\dot{c}/c = \gamma = \frac{\dot{k}}{k} = \gamma_k$ . This leaves one more growth rate to go: the growth rate of

human capital,  $\frac{\dot{h}}{h}$  ( $\equiv \gamma_h$ ).

Take (7) and move all the constants to the left hand side. We obtain:

$$\frac{(\gamma\sigma + \rho)}{A\beta} = k^{-(1-\beta)} u^{1-\beta} h^{1+\psi-\beta} \quad (11)$$

If we take logs and derivatives of both sides the LHS will be zero, of course, and we get:

$$0 = -(1-\beta)\frac{\dot{k}}{k} + (1+\psi-\beta)\frac{\dot{h}}{h} \quad (12)$$

which we can re-arrange to obtain:

$$\frac{\dot{h}}{h} \equiv \gamma_H = \gamma \frac{1-\beta}{1+\psi-\beta} \quad (13)$$

from which we can see that, in the absence of the human capital externality,  $\gamma = \gamma_h$ .

Now we must determine the value of  $\gamma$  or  $\gamma_h$  as a function of the parameters of the model. From (6ii) we can write

$$\begin{aligned} \frac{v}{\lambda} &= \phi h [A(1-\beta)k^\beta u^{-\beta} h^{1+\psi-\beta}]^{-1} \\ &= \phi [A(1-\beta)k^\beta u^{-\beta} h^{\psi-\beta}]^{-1} \end{aligned} \quad (14)$$

Now, once again, take logs and derivatives of both sides to get

$$\frac{\dot{v}}{v} = \frac{\dot{\lambda}}{\lambda} - \beta \frac{\dot{k}}{k} - (\psi-\beta)\frac{\dot{h}}{h} \Rightarrow \frac{\dot{v}}{v} + \beta\gamma + (\psi-\beta)\gamma_h = \frac{\dot{\lambda}}{\lambda} \quad (15)$$

We can easily get an expression for  $\dot{v}/v$  from (6iii):

$$\frac{\dot{v}}{v} = -(\beta A k^{\beta-1} u^{1-\beta} h^{1+\psi-\beta}) = -(\gamma\sigma + \rho) \quad (16)$$

where the last equality follows from (9).

Now to find the value of  $\dot{\lambda}/\lambda$  divide both sides (6iv) by  $\lambda$ , and then substitute for  $v/\lambda$  from (12), and we obtain:

$$\begin{aligned}\frac{\dot{\lambda}}{\lambda} &= -\phi[A(1-\beta)k^\beta u^{-\beta} h^{\psi-\beta}]^{-1}[A(1-\beta)k^\beta u^{1-\beta} h^{\psi-\beta}] - \phi(1-u) \\ &= -\phi u - \phi(1-u) \\ &= -\phi\end{aligned}\tag{17}$$

which means that the shadow price of human capital is decreasing at the constant rate,  $\phi$ , which is the productivity parameter of the "studying technology."

Finally, we substitute (10) and (11) into (9) and use the result from (7) to substitute for  $\gamma$  and we obtain:

$$\gamma_h = \frac{(\phi - \rho)(1 - \beta)}{\sigma(1 + \psi - \beta) - \psi}\tag{18}$$

Notice that if there is no human capital externality,  $\psi = 0$ , and thus we find that  $\gamma = \gamma_h = (\phi - \rho)/\sigma$ . With  $\phi = A$  this is the same growth rate as in the Rebelo model. Note the implication; if the production of human capital improves (i.e.,  $\phi$  increases) the growth rate increases. So the productivity of human capital accumulation affects growth. The policy implications are clear.

When the human capital externality is present the competitive solution differs from the command optimum. The planners' problem would internalize the externality, since now the effect of the choice of  $h$  on  $h_a$  would be taken into account. Rewrite the Hamiltonian and perform the usual substitutions to derive:

$$\gamma_h = \sigma^{-1} \left( \phi - \frac{(1-\beta)\rho}{1+\psi-\beta} \right)\tag{19}$$

which is higher than the market solution, if  $\sigma^{-1}$  is not too big. If the elasticity of substitution ( $\sigma^{-1}$ ) is too big then agents are unwilling to defer consumption, and this overcomes the external benefits that the planner is taking into account. When  $\sigma^{-1}$  is not too big then the private return to studying is lower than the social return, so agents (in the decentralized solution) will not invest in human capital as much as would be socially optimal.

## A Model With Public Goods (Barro)

Barro considers a model where public expenditure is productive. It is easy to think of investments in infrastructure that make private production more profitable.

Let  $G$  be aggregate services, then  $g = G/N$  is the quantity allocated to each of  $n$  producers. Notice that this is not exactly what we usually think of as a public good, since it is rival and excludable. We could think of infrastructure such as phone lines or roads to factories. In any event, this allows us to write the production function:

$$y = Ak^{1-\alpha}g^\alpha \quad (1)$$

Notice that  $y$  is subject to diminishing returns to  $k$ , but not to  $k$  and  $g$ . The individual producer takes  $g$  as fixed (i.e., independent of his decision about  $k$ ).

The government runs a balanced budget; hence  $\tau = g/y$ . Since  $g$  uses one unit of the single output good, efficiency requires  $g^*$  such that  $\partial y/\partial g^* = 1$ . Now if  $g$  is set efficiently, then from (1) it follows that  $g/y = \alpha$ . This follows because

$$\begin{aligned} \frac{\partial y}{\partial g} &= \alpha Ak^{1-\alpha}g^{\alpha-1} \\ \text{so} \\ 1 &= \alpha yg^{-1} \Rightarrow \frac{g}{y} = \alpha \end{aligned} \quad (2)$$

Now the marginal product of capital, determined from (1) is

$$\begin{aligned} \frac{\partial y}{\partial k} &= (1-\alpha)A \frac{1}{k^\alpha} \left(\frac{g}{y}\right)^{\frac{\alpha}{1-\alpha}} \\ &= (1-\alpha)A \frac{1}{k^\alpha} \alpha^{\frac{\alpha}{1-\alpha}} \end{aligned} \quad (3)$$

where the last equality follows if  $g = g^*$ .

The private return to investment is what is left after taxes:

$$(1-\tau)\frac{\partial y}{\partial k} = (1-\tau)\left[(1-\alpha)A \frac{1}{k^\alpha} \left(\frac{g}{y}\right)^{\frac{\alpha}{1-\alpha}}\right] \quad (4)$$

We can substitute the RHS of (4) into the Rebelo equation to solve for the growth rate of the economy.

Notice that with  $\tau = 0$  social and private returns are equal. Otherwise, the private return to investment is less than the social, because entrepreneurs do not consider the effect they have on others through investment. The



channel is that with higher investment, and thus income, there is more government spending, which, since it is productive, makes for higher growth. But individual investors do not take into account the effect on  $y$  from their investments.

## **Concluding Note**

These notes are an introduction to the literature on endogenous growth models. This literature is exploding, so it is not possible to be comprehensive. The purpose of these notes is to develop the basic structure of these models, and to make it clear how they obtain results so different from exogenous growth models.

There is one important lacuna in these notes. I have not covered models that generate endogenous growth via the introduction of new products. This literature is important, and its absence here should not be taken to imply any judgment about these models. I hope to remedy this defect in the next edition of these notes.

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